Quadratics

Things you should already know

Fact (Completing the Square) —

$$x^2 + 2bx + c = (x+b)^2 + c - b^2$$

Fact (Quadratic Formula) —

$$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Fact (The Discriminant) —

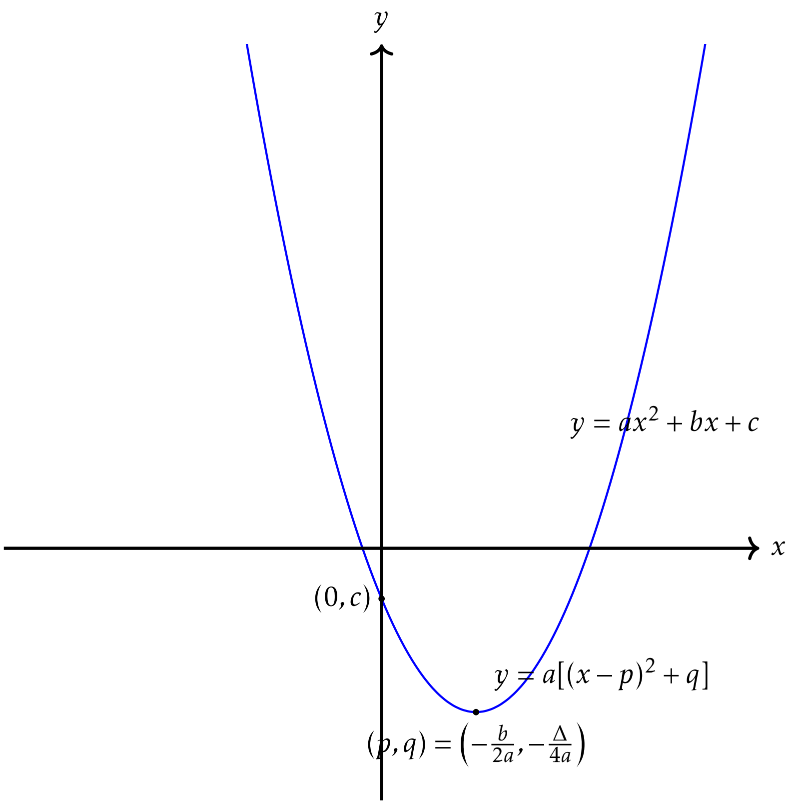
$$\Delta = D = b^2 - 4ac$$

$D > 0 \Rightarrow 2$ *distinct* real roots

- $D = 0 \Rightarrow 1$ *repeated* real root
- $D < 0 \Rightarrow 0$ real roots

Fact (Squares are Non-negative) —

$$x^2 \geq 0$$



Example

Prove that:

(i) if $a + 2b + 3c = 7x$, then

$$a^2 + b^2 + c^2 = (x - a)^2 + (2x - b)^2 + (3x - c)^2;$$

(ii) if $2a + 3b + 3c = 11x$, then

$$a^2 + b^2 + c^2 = (2x - a)^2 + (3x - b)^2 + (3x - c)^2.$$

Give a general result of which **(i)** and **(ii)** are special cases.

(i)

$$\begin{aligned}(x-a)^2 + (2x-b)^2 + (3x-c)^2 &= x^2 - 2ax + a^2 + 4x^2 - 4bx + b^2 + 9x^2 - 6cx + c^2 \\ &= (1^2 + 2^2 + 3^2)x^2 - 2x(a + 2b + 3c) + a^2 + b^2 + c^2 \\ &= 14x^2 - 2x(7x) + a^2 + b^2 + c^2 \\ &= a^2 + b^2 + c^2\end{aligned}$$

(ii)

$$\begin{aligned}(2x-a)^2 + (3x-b)^2 + (3x-c)^2 &= (2^2 + 3^2 + 3^2)x^2 - 2x(2a + 3b + 3c) + (a^2 + b^2 + c^2) \\ &= 22x^2 - 2x(11x) + a^2 + b^2 + c^2 \\ &= a^2 + b^2 + c^2\end{aligned}$$

The general result is:
If $\frac{A^2+B^2+C^2}{2}x = Aa + Bb + Cc$ *then* $(Ax - a)^2 + (Bx - b)^2 + (Cx - c)^2 = a^2 + b^2 + c^2$
Alternatively, if $\lambda = \frac{2\mathbf{x}\cdot\mathbf{y}}{||x||^2}$ *then* $||\lambda\mathbf{x} - \mathbf{y}||^2 = ||\mathbf{y}||^2$ *which is easy to see is true.*

Example

Find the range of values for k for which the equation

$$x^2 - kx + (k+3) = 0 \quad \text{has real roots}$$

For $x^2 - kx + (k+3) = 0$ to have a real root, the discriminant must be non-negative, ie

$$\begin{aligned} 0 &\leq \Delta \\ 0 &\leq (-k)^2 - 4 \cdot 1 \cdot (k+3) \\ &= k^2 - 4k - 12 \\ &= (k-6)(k+2) \end{aligned}$$

We draw a quick sketch to establish:
$k \leq -2$ or $k \geq 6$
We could write this as $\{k : k \leq -2\} \cup \{k : k \geq 6\}$

Example

Find the range of the function

$$g(x) = \frac{x^2 - 4x + 3}{x^2 - 6x + 10}$$

Suppose $g(x) = k$, then

$$\begin{aligned} & k = \frac{x^2 - 4x + 3}{x^2 - 6x + 10} \\ \Rightarrow \quad & 0 = (1-k)x^2 + (6k-4)x + (3-10k) \\ \Rightarrow \quad & 0 \leq \Delta \\ & = (6k-4)^2 - 4(1-k)(3-10k) \\ & = 4(3k-2)^2 - 4(1-k)(3-10k) \\ & = 4((9k^2 - 12k + 4 - (3 - 13k + 10k^2))) \\ & = 4(-k^2 + k + 1) \end{aligned}$$

Therefore $k \in \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$

Example

Solve $x = \sqrt{x} + 12$

Method 1:

$$\begin{aligned} & & x &= \sqrt{x} + 12 \\ &\implies & x - 12 &= \sqrt{x} \\ &\implies & (x-12)^2 &= x \\ &\implies & x^2 - 24x + 144 &= x \\ &\implies & x^2 - 25x + 144 &= 0 \\ &\implies & (x-9)(x-16) &= 0 \\ &\implies & x &= 9, 16 \end{aligned}$$

But we might notice $\sqrt{9} + 12 = 15 \neq 9$*(!).*

Method 2:

$$\begin{aligned} & & x &= \sqrt{x} + 12 \\ y = \sqrt{x}: & & y^2 &= y + 12 \\ &\implies & y^2 - y - 12 &= 0 \\ &\implies & (y-4)(y+3) &= 0 \\ &\implies & y &= 4, -3 \\ &\implies & \sqrt{x} &= 4, -3 \\ &\implies & x &= 16 \end{aligned}$$